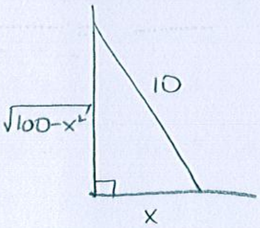
A 10 ft long ladder is leaning against a wall. The base of the ladder is being pushed towards the wall at 2 ft per second. How quickly is the area between the ladder, the wall and the ground changing when the base of the ladder is 6 ft from the wall ? **SCORE: _____ / 25 PTS**

NOTE: Give the units of your final answer. State clearly whether the area is growing or shrinking.



$$\frac{dx}{dt} = \frac{-2\text{ft}}{\text{sec}}$$

WANT $\frac{dA}{dt}$ WHEN $x = 6\text{ft}$

WHERE A = AREA UNDER LADDER

$$A = \frac{1}{2}x\sqrt{100-x^2}$$

$$\frac{dA}{dt} = \left(\frac{1}{2}\sqrt{100-x^2} + \frac{1}{2}x\left(\frac{1}{2\sqrt{100-x^2}}\right)(-2x)\right)\frac{dx}{dt}$$

$$= \left(\frac{1}{2}(8) + \frac{1}{2}(6)\frac{1}{2(8)}(-2(6))\right)(-2)$$

$$= \left(4 + 3\left(\frac{1}{16}\right)(-12)\right)(-2)$$

$$= -\frac{7}{2}\ \text{ft}^2/\text{sec}$$

THE AREA IS SHRINKING

Find the slope of the tangent line to the curve $8+\cot\frac{\pi}{y}=x^4y^2+7x$ at $(-1, 4)$. **SCORE: _____ / 20 PTS**

$$\left(-\csc^2\frac{\pi}{y}\right)\left(-\frac{\pi}{y^2}\frac{dy}{dx}\right)=4x^3y^2+2x^4y\frac{dy}{dx}+7$$

$$\left(-\csc^2\frac{\pi}{4}\right)\left(-\frac{\pi}{16}\left.\frac{dy}{dx}\right|_{(-1,4)}\right)=4(-1)(16)+2(1)(4)\left.\frac{dy}{dx}\right|_{(-1,4)}+7$$

$$\frac{\pi}{8}\left.\frac{dy}{dx}\right|_{(-1,4)}=8\left.\frac{dy}{dx}\right|_{(-1,4)}-57$$

$$\left.\frac{dy}{dx}\right|_{(-1,4)}=\frac{57}{8-\frac{\pi}{8}}$$

If $f(x) = (3^x + 1)^{-\sec x}$, find the **equation of the normal line** at the point where $x = 0$. SCORE: _____ / 25 PTS

$y = (3^x + 1)^{-\sec x} \longrightarrow x = 0 \longrightarrow y = 2^{-1} = \frac{1}{2}$

$\ln y = -\sec x \ln(3^x + 1)$

$\frac{1}{y}\frac{dy}{dx} = (-\sec x \tan x)\ln(3^x + 1) - \sec x \left(\frac{1}{3^x + 1}\right)(3^x \ln 3)$

$2 \left.\frac{dy}{dx}\right|_{x=0} = (-(1)(0))\ln 2 - (1)(\tfrac{1}{2})(\ln 3)$

$= -\frac{1}{2}\ln 3$

$\left.\frac{dy}{dx}\right|_{x=0} = -\frac{1}{4}\ln 3$

SLOPE OF NORMAL $= \frac{4}{\ln 3}$

$y - \frac{1}{2} = \frac{4}{\ln 3}x$

Let $f(x) = \dfrac{\arcsin x}{x^2}$.

SCORE: _____ / 25 PTS

[a] If x changes from 0.5 to 0.4, find dy.

$$f'(x) = \frac{\frac{1}{\sqrt{1-x^2}}(x^2) - (\arcsin x)(2x)}{x^4}$$

$$dx = \Delta x = 0.4 - 0.5 = -0.1$$

$$f'(\tfrac{1}{2}) = \frac{\frac{1}{\sqrt{1-\frac{1}{4}}}(\frac{1}{4}) - (\arcsin \frac{1}{2})(1)}{(\frac{1}{2})^4}$$

$$= \frac{\frac{\cancel{2}}{\sqrt{3}} \cdot \frac{1}{\cancel{4}_2} - \frac{\pi}{6}}{\frac{1}{16}}$$

$$= 16\left(\frac{\sqrt{3}}{6} - \frac{\pi}{6}\right) = \frac{8}{3}(\sqrt{3} - \pi)$$

$$dy = \frac{8}{3}(\sqrt{3} - \pi)(-\tfrac{1}{10})$$

$$= -\frac{4}{15}(\sqrt{3} - \pi) = \frac{4}{15}(\pi - \sqrt{3})$$

[b] Approximate $f(0.4)$ **using your answer to part [a]**.

$$f(\tfrac{1}{2}) = \frac{\frac{\pi}{6}}{\frac{1}{4}} = \frac{2\pi}{3}$$

$$f(0.4) \approx \frac{2\pi}{3} + \frac{4}{15}(\pi - \sqrt{3}) = \frac{14\pi}{15} - \frac{4\sqrt{3}}{15}$$

If $f(x) = \dfrac{g(x^4)}{x}$, find a formula for $f''(x)$. Your answer may involve g, g' and/or g''. **SCORE: _____ / 20 PTS**

$$f(x) = x^{-1} g(x^4)$$

$$f'(x) = -x^{-2} g(x^4) + x^{-1} g'(x^4)(4x^3)$$

$$= -x^{-2} g(x^4) + 4x^2 g'(x^4)$$

$$f''(x) = 2x^{-3} g(x^4) - x^{-2} g'(x^4)(4x^3) + 8x g'(x^4) + 4x^2 g''(x^4)(4x^3)$$

$$= 2x^{-3} g(x^4) + 4x g'(x^4) + 16x^5 g''(x^4)$$

ALTERNATE SOLUTION BELOW

$$f'(x) = \frac{g'(x^4)(4x^3)x - g(x^4)(1)}{x^2} = \frac{4x^4 g'(x^4) - g(x^4)}{x^2}$$

$$f''(x) = \frac{[16x^3 g'(x^4) + 4x^4 g''(x^4)(4x^3) - g'(x^4)(4x^3)]x^2 - [4x^4 g'(x^4) - g(x^4)](2x}{(x^2)^2}$$

$$= \frac{12x^5 g'(x^4) + 16x^9 g''(x^4) - 8x^5 g'(x^4) + 2x g(x^4)}{x^4}$$

$$= \frac{16x^8 g''(x^4) + 4x^4 g'(x^4) + 2g(x^4)}{x^3}$$

The position of an object (in meters) at time t (in minutes) is given by the function $s(t) = \dfrac{(2t+1)^2}{\sqrt[4]{t}}$ **SCORE: _____ / 20 PTS**

for $t \geq 0.5$. Find the **acceleration** of the object at time $t = 1$. **Give the units of your final answer.**

$$s(t) = \frac{4t^2+4t+1}{t^{\frac{1}{4}}} = 4t^{\frac{7}{4}} + 4t^{\frac{3}{4}} + t^{-\frac{1}{4}}$$

$$s'(t) = 7t^{\frac{3}{4}} + 3t^{-\frac{1}{4}} - \frac{1}{4}t^{-\frac{5}{4}}$$

$$s''(t) = \frac{21}{4}t^{-\frac{1}{4}} - \frac{3}{4}t^{-\frac{5}{4}} + \frac{5}{16}t^{-\frac{9}{4}}$$

$$s''(1) = \frac{21}{4} - \frac{3}{4} + \frac{5}{16}$$

$$= \frac{84-12+5}{16} = \frac{77}{16}\ \frac{m}{min^2}$$

Prove the derivative of $\tan^{-1} x$.

SCORE: _____ / 15 PTS

HINT: Consider the proof of the derivative of $\sin^{-1} x$ that was shown in lecture.

$$y = \tan^{-1} x$$

$$x = \tan y$$

$$1 = \sec^2 y \frac{dy}{dx}$$

$$\frac{dy}{dx} = \frac{1}{\sec^2 y} = \frac{1}{1 + \tan^2 y} = \frac{1}{1 + x^2}$$